Enliven Archive Intensifying Thoughts

JOURNAL OF MATHEMATICS, STATISTICS AND COMPUTING

OPEN ACCESS

Research Article

History of Indian Mathematics

A Brief Review in the Perspective of Ancient and Medieval periods

Choudhury KD*

Department of Mathematics, Assam University, Silchar, India.

***Corresponding Author:**

Karabi Dutta Choudhury,
Department of Mathematics, Assam University, Silchar, India.
Email: karabidc@gmail.com

Received: January 10, 2020
Accepted: January 20, 2020
Published: January 25, 2020

Abstract

Indian mathematics has its glorious root in the civilization. This article explores the various causes of lacking of its expansion during ancient and medieval periods. Its pedagogical advantages in teaching are well known and, therefore the advance research in history of Indian mathematics may only explore many unknowing facts in the field of Mathematics.

Keywords: History of Indian Mathematics; Ancient and Medieval Periods; Pedagogical Advantages.

Introduction

The history of mathematics is a vast topic and can never be studied completely as the mathematical works of the ancient times remain undiscovered or has been lost through time. Mathematics on the Indian subcontinent has a rich history going back over 5000 years and significant contributions were made long before similar advances were made in Europe and other western countries. But today the western contributions are known more than that of the Indian. The reason behind that is the lack of linguistic co-ordination between India and other part of the world. The linguistic challenges are compounded by the fact that the earliest known Indian languages are part of oral tradition rather than a written one.

The history of mathematics reveals the great works of ancient Indian mathematicians, including even the mathematical contents of the Vedas. The importance of the great mathematical achievements in ancient India cannot be ignored, rather the fact remains that as soon as a new mathematical work originates, and its history begins along with it and keeps taking shape as the subject develops.

Apart from Indians, the whole world feels proud of the importance of the great mathematical achievements in ancient India. Thus the history of mathematics cannot be completed without the history of ancient Indian mathematics.

The studies of the history of mathematics in India have not been in proper way, because of the following reasons as described by **Vinod Mishra** (2011)[14] in his work entitled "*MATHEMATICAL HERITAGE OF INDIA: SOME REMEDIAL ISSUES: MATHEMATICS HISTORY IN TEACHING*"

1) The proofs were not explicitly mentioned through they were understood well.
2) Chronological order had not been strictly adhered to.
3) There was no tradition of educating en masses through proper writings
4) Till the middle of eighteenth century A.D. Europeans were unaware of Indian culture and its contributions, though they knew it little from various sources

The objective of the present work is to highlight significant, positive, and concrete contributions made by ancient Indian mathematicians in the initial advancement of mathematics and possibly relate them to the developments elsewhere in the world in those days, particularly to those in Greece, the Middle East, China, and Japan.

In the work of Dharampal (1971) [3], Needham (1954) [2], and Van Sertima (1983) [6] and summarised by Teresi (2002) [12], it seems to indicate the existence of scientific creativity and technological advancements long before the incursions of Europe into those areas. If this is true, we need to understand the dynamics of pre-colonial science and technology in those countries which gave rise to these developments and to find meaningful ways of adapting to present-day requirements the indigenous and technological forms that still remain. The main objective of this article explores the glorious development of the ancient mathematics along with the approximate time period and the different causes for lacking of its expansion during ancient and medieval periods.

Ancient Indian Mathematics

First phase of Indian Mathematics (Indus valley-civilization)

The earliest known urban cultured civilization was at Harappa near Punjab and in Mohenjo- Daro which was near the Indus river.

The Indus Valley civilization or the Harappan civilization, which is said to be the earliest civilization of the Indian subcontinent, also known as the bronze age of Indian civilization. But there is a confusion regarding the beginning of this civilization. When John Marshell excavated the Indus valley in 1922, he gave it the date of about 3000 BC, while Mortimer Wheeler put the Indus valley civilization between 2450BC to 1900BC. But evidences of religious practices in this area are date back around 5500 BC. Again, according to the paper published in “Nature” journal on May 25, 2016, scientists from IIT Kharagpur and ASI have uncovered evidences that this civilization is 8000 years old.

Thus, it is difficult to draw the graph of the time space of first phase of Indian Mathematics but the mathematical activities during this period are clear from the fact that the basic mathematics was in use at that time, which is evident from the various designs on the cravings which had regular geometric shapes like concentric circles and triangles etc. The use of circles in the wheels, which had a metallic band wrapped around the rim found in the pictures of the Bullock cart clearly indicates to the fact about knowledge of the ratio of circumference of the circle and its diameter, and hence the value of pi [12].

The entire Indus valley civilization has many sub periods, which are given below:

1. Pre Harappan period (7000BC-3300BC)
2. Early Harappan period (3300BC-2600BC)
3. Mature Harappan period (2600BC-1900BC)
4. Late Harappan period (1900BC-1300BC)
5. Post Harappan period (1300BC-300BC)

In Indian subcontinent, many movements and invasions of people occurred from outside during this period. So it is difficult to get the written proofs or evidences of mathematical activities during this period. But excavations laid in Mohenjo - Daro and Harappa and the neighboring Indus river valley gave evidence of practical applications of basic mathematics during that period ,i.e the people during that period have great knowledge of weights and measuring scales and a basic geometry was also known to them. Besides these, the sculpture of those cities indicates the technological advancement during that period of Indus- Valley Civilization.

Some of them are mentioned below:

1. According to an analysis of Harappan weights and measures, decimal system was in use during the Harappan period. Weights corresponding to the ratios of 0.05, 0.1, 0.2, 0.5, 1, 2, 5, 10, 100, 200 and 500 have been identified.(Archaeology online)

2. Some length measuring appliances were also in use during that period such as:

a) A remarkably accurate decimal ruler known as Mohenjo- Daro ruler was found. Its sub division has a maximum error of 0.05 inches at a length of 1.32 inches. This length is known as the Indus length. (Sykorova).

b) Another scale was discovered, when a bronze rod was found which was marked up to 0.367 inches, which was considered to be highly accurate during that time. The 100 units of this measure is 3.67 inches which is the measure of a stride. (Sykorova)

Such scales were implemented in town - planning like road construction, drainage system, construction of homes with proper precession.

3. During Indus valley civilization, the following field of Mathematics was developed viz.

i) Mathematical astronomy
ii) Arithmetic, algebra
iii) Trigonometry
iv) Geometry and
v) Combinatorics

According to the British Archaeologists Mortimer Wheeler, Indo-Aryans coming from the Central Asia conquered the Indus river valley and thus the great civilization ended. But it is also said that the Indus civilization ended due to dramatic climatic changes.

B. Vedic period (2000-1000 BC)
C. Sulba Sutras (1000-500 BC)

The Vedic period or the Vedic age, also known as the “iron age” is the period in the history of the Indian mathematics intervening between the urban Indus Valley civilization and a second civilization, which began when groups of Indo-Aryan people migrated into the northwest India and started to inhabit the northern Indus valley in 1500BC. They were the founder of Vedic religion. Their works are the first literary evidence of Indian culture including Mathematics [1].

In the Vedic period, records of mathematical activities are found in Vedic texts associated with ritual practices [9].

The word Vedic comes from the collection of sacred texts known as Vedas. There are 4 Vedas:

a) Rigveda: It consists of 1028 hymns divided into 8 or 10 books.
b) Atharvaveda: The word Atharva literally means “Fire priest”. It contains large number of magical formulae and contains definite pre Vedic influences.
c) Samaveda: It contains large number of Vedic mantras. These mantras used to be sung and there are instructions for the tunes.
d) Yajurveda: It mainly deals with the works of the sacrifice.

The entire Vedic period is divided into sub-periods:

EARLY VEDIC PERIOD. (1700BC -1000BC)

The Vedic texts reveal that a well-established decimal number system was used to express quantities during that period. No mathematical texts are found from this period as in those days information were transferred orally and was recited number of times so that they were well remembered. But still they used mathematics on a large basis for carrying out their rituals. Surprisingly they used big numbers as powers of ten from hundred upto a trillion. The Mantras found in the rigveda involves powers of ten from hundred to as large upto trillion which signifies the use of the arithmetic operations (ganit) such as addition, subtraction, multiplication, fraction, squares, cubes, roots. These were first found in yajurveda’s description of sacrificial texts.

LATER VEDIC PERIOD(1000BC-500BC)

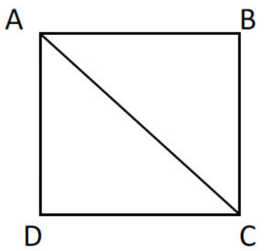
This phase of the Vedic period is also known as the sulba sutra period as the first significant chapter of **sulba sutra** was written by ***Baudhyayana*** around 800BC [8].

The shape of the fire altars used for various religious practices involves certain geometrical shapes like circles, triangles, rectangles, rhomboids etc. One shape was required to change to another shape, keeping area fixed. Methods for transforming a square into rectangle, trapezium, rhombus, and triangle and vice versa was found in the **BAUDHYAYANA SULBA SUTRAS**.

In this process, it is recognized that the square on the diagonal of a given square contains twice the original area.

If ABCD be a square of side of measure 'l', say then the area of the square ABCD is given by l^2.

Then as per the process given in the **BAUDHYAYANA SULBA SUTRAS**



$AC^2 = 2l^2$

$= l^2 + l^2$

$= AB^2 + BC^2$

Or $AC^2 = AD^2 + CD^2$

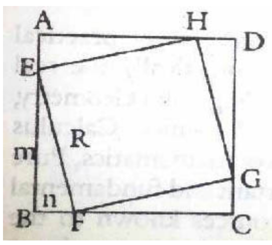
which is the well known ***Pythagoras theorem***.

The Pythagoras theorem was known to ancient Indians long before the time of Pythagoras. It is found that even Pythagoras learnt his basic geometry from the Vedas and the Vedic Sutras.

This theorem has a vast application in the practical mathematical works and is acknowledged by all as it is basic requirement for higher geometry including Solid and Spherical Geometry, Calculus, Trigonometry etc. and many other branches of both pure and applied mathematics.

There are several Vedic proofs of this theorem, which are more easier than the Euclid's proof.

One of the proof is described below:-



Let us consider two squares ABCD and EFGH one inside the other, as shown in the figure.

Here AE = BF = CG = DH = n; EB = FC = GD = HA = m;

Let the side of the square EFGH be 'h'.

Thus, AB=m+n

Now, the square EFGH + the four congruent right-angled triangles around it = the square ABCD

$\therefore h^2 + 4\,(1/2\ mn) = (m + n)^2$

Which is on simplifying, $h^2 = m^2 + n^2$

This is the result of the ***Pythagoras theorem***.

Some of such constructions are described below:

Suppose a square of side equal to the length 'a' is given. A cord with length equal to the side of the square, 'a' is increased to a length of '(3/2) a' and at a distance of '(5/12)a', a mark is made from one end. When the end points are fixed at a distance 'a' apart, pulling the line downwards creates a 5-12-13 right angled triangle.

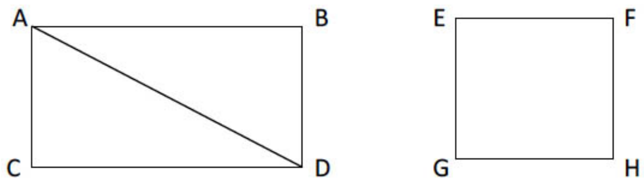
AB = a, AC = 5/12 a, BC = (3/2 a -5/12 a) = 13/12 a

Thus we get from the figure

$AB^2 + AC^2 = a^2 + (5/12)^2\,a^2 = (13/12)^2\,a^2 = BC^2$

Which is Pythagoras Theorem.

Moreover areas involving multiples of 3 were also constructed as follows: Let us take a square EFGH of side 's' and a rectangle ABCD is made with width equal to 's' and length equal to root of 2s, which is called the doubler. Then the diagonal of the rectangle is said to be a tripler, which produces a square of three times the original area.



Then using pythagoras theorem, we have in triangle ABD

$AD^2=s^2 + 2s^2 = 3s^2$

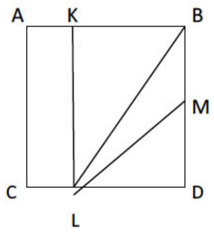
Thus $1/3\ s^2 = 1/9\ AD^2$

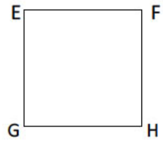
In other words, one-third of the original area will be one-ninth of the square of the tripler.

Another transformation from one figure to another is done by adding or subtracting square areas were also used.

The method for adding and subtracting two squares of different sizes depend on the relations between the sides and hypotenuse of triangles.

Let ABCD be a square of larger area and EFGH be square of smaller area. We cut off a rectangle whose width is equal to the side of the shorter square and length equal to the side of the larger one.





Then the diagonal LB will be the side of a square equal to the sum of the two given squares. But f the side KL is placed diagonally as the segment LM, then the cut-off side MD is the side of a square equal to the difference of the given squares.

During Vedic period in the following field, Mathematics was developed

i) Astronomy and Geometry
ii) Estimation of value of π
iii) Early forms of Pytha goras theorem
iv) Four arithmetical operators
v) Arithmetic sequence
vi) Fractions found in Samihitas

During Later Vedic period in the following field, the Mathematics was developed

i) Irrational Numbers
ii) Quadratic Equations
iii) Pythagorean triplets

PANINI (6^{th} - 4^{th} century BC)

We have found that all the earliest known mathematical works and treatises are composed in verses in Sanskrit, so here it is necessary to put some light on the introduction of the Sanskrit language in the Indian history. G G Joseph(2000) [10], in the Crest of the Peacock argued that the algebraic nature of the Indian mathematics arises as a consequence of the structure of the Sanskrit language.

Panini, who was well-known as the father of linguistic, was an ancient Sanskrit linguistic and grammarian. His grammar formed the foundation of rigorous intellectual work in India over 2 million years. He lived in 5^{th} or 6^{th} century BC, while other estimates his flourishing period before mid 4^{th} - century BC. As Panini's grammar defines classical Sanskrit, so Panini is chronologically placed in the later part of Vedic period, during the Mahajanapada era (6^{th} - 4^{th} century BC).

The pioneering work by Panini in the field of Sanskrit grammar and linguistic was an important development in the history of Indian science and technology, which had a great impact on the mathematical treatises. Today, Panini's constructions can also be seen as comparable to modern definitions of a mathematical function.

The existence of zero was found in the work of the famous Indian grammarian Panini (5^{th} or 6^{th} century BC) and later the work of Pingala a scholar who wrote a work Chhandas - Sutra (c. 200 BC).

The first documented evidence of the use of zero for mathematical purposes is not until around 2nd century AD (in the Bakhshali manuscript).

Some applications of Mathematics by the Vedic Rules

Highest Common Factor

There are two ways of finding a H .C.F in the current system of mathematics.

The first m ethod is by factorization, which is difficult and time consuming.

The second method is a process of continuous division, which is a mechanical process and can be applied in all the cases. But it is too mechanical as well as long and crumby.

The third one is the Vedic method provides a method which is free from the above disadvantages. Again the Vedic rule consists of two sub-rules for finding the H.C.F as follows:-

1. The Adyamadyena rule is to find the factors of the expressions and name the common factor as the H.C.F.

2. The Lopa-Sthapana-Sutra, which means the addition or subtraction to eliminate and retain the highest power of the dependent terms.

The process of finding the H.C.F by Lopa-Sthapana process is described below:-

Suppose we have to find the H.C.F of

A= $(x^2 + 7x + 6)$ and
B= $(x^2 + 5x + 6)$

Then for eliminating the highest power, we subtract B from A; and to eliminate the lowest one,we add the two expressions.

Subtraction :- $(x2 + 7x + 6) - (x2 - 5x - 6) + 12(x + 1)$
Addition :- $(x2 - 5x - 6) + (x2 + 7x + 6) = 2x(x + 1)$

We then remove the common factor if any from each and we find (x+1) as the H.C.F.

Multiple Simultaneous Equations

We will discuss the method for solving the simultaneous equations involving three or more unknowns.| The sutras namely, Lopa-Sthapana Sutra, the Anurupya Sutra and the Paravartya sutra are used for solving the simultaneous equations.

In the first type,

We have a significant figure on the R.H.S of only one equation and zeroes on the other two equations. We derive a new equation from the homogeneous zero equation, defining any two of the unknowns in terms of the third unknown. We then substitute the values in the third equation.

1) The first method is described as follows:-
We consider three equations.

$x+y-z=0$ ....(A)
$4x-5y+2z=0$ ....(B)
$3x+5y+z=10$ ....(C)

Then, (A)+(C) gives $4x+6y=10$
2(A)+(B) gives $6x-3y=0$

Solving the above new equations, we get 10x=10.

Thus, x=1 y=2 z=3

2) The second method involves the addition and subtraction of proportionate multiples for elimination of one unknown and the retention of the other two.

From (A), we have x+y=z
From (B), we have 4x-5y=-2z
Thus, by ***Parvatya***, x= z/3 , y=2z/3

By substituting x and y in C, we get, *z+1(1/3)z+z=10*

In the second type,

There is a significant figure in the R.H.S of all the three equations. It can be solved by Parvatya cross-multiplication method to produce two equations whose R.H.S contains zero only or by the processes mentioned above.

We consider the simultaneous equations as:-

x+2y+3z=14..............(A)
2x+3y+4z=20............(B)
3x+y+6z=23...............(C)

14*(B) gives, 28x+42y+56z=280
20*(A) gives, 20x+40y+60z=280

Therefore, 8x+2y-4z=0..................(D)

(C)*14 gives, 42x+14y+84z=322
(A)*23 gives 23x+46y+69z=322

Therefore, 19x-32y+15z=0...............(E)

Now, we apply the first method to solve the equations (D) and (E) and get x=1, y=2, z=3. Again by Lopa-Sthapana method, we say,

2A-B gives, y+2z = 8
3A-C gives, 5y+3z = 19

Therefore, x = 1, y = 2, z = 3.

Jain Mathematics: (300BC-400AD)

Jain mathematics is a crucial link between the mathematics of the Vedic period and that of the classical period. Jain mathematicians made a significant contribution by removing religious and ritualistic constraints from the Indian mathematics. They had divided the large numbers and infinites into three categories:- enumerable, innumerable and infinite. They also described five types of infinity as

i) the infinite in one direction,
ii) the infinite in two direction,
iii) the infinite in area,
iv) the infinite everywhere ,and
v) the infinite perpetually.

The mathematicians flourished during this period and their works are:

Bhadrabahu (298BC): He is the author of two astronomical works namely, Bhadrabahavi- Samhita, and a commentary on the Surya-Prajnapati,

Yativrisham Acharya (176BC): He authored a mathematical text called, Tiloyapannati.

Umasvati (150BC): He is better known for his Jain philosophy and metaphysics, and authored a mathematical work called Tattawarthadhigama-Sutra.

Other Jain mathematical works include:

Vaishali Ganit (3rd century BC), Sthananga Sutra (300BC-200BC), Anoyogdwar (200BC-100BC), Satkhandagama (2nd century BC)

Mahavira (or Mahaviracharya), a Jain by religion, is the most celebrated Indian mathematician of the 9^{th} century.

Mahavira was one of the great Jain mathematician, lived in Karnataka, India. His highly respected works include establishment of terminology for concepts such as Equilateral, isosceles triangles, rhombus, circle and semi circle. He was the author of a book namely, Ganita sara sangraha, which is a generalization of Brahma Sphuta Siddhanta, which is one of the earliest mathematical texts. This book gave systematic rules for expressing a fraction as the sum of unit fractions, which follows the use of unit factors in Indian mathematics in the Vedic period.

▪ He emphasized on the development of techniques for solving problems on algebra. He discovered the algebraic identities,

$a^3 = a(a + b)(a - b) + b^2 (a - b) + b^3$

▪ He found out the formula for nCr as,

n(n-1)(n-2).......(n-r-1)/r(r-1)(r-2).....2.1

▪ He devised a formula which approximated the area and perimeters of ellipses and found the methods to calculate the square of a number and cube roots of a number. He also stated that the square root of negative numbers does not exist.

The second section of the books devoted to arithmetic named as: Kala-Savara-Vyavahara. It highlights the following rules:

▪ To express 1 as a unit of n unit fractions.
▪ To express 1 as a sum of odd number of unit fractions.
▪ To express a unit fraction 1/ q as the sum of n other fractions with given numerators, say, 1 2 3 . , , ,...... n a a a a .
▪ To express p/q as the sum of unit fractions.
▪ To express a unit fraction as the sum of two unit fraction.
▪ To express p/q as the sum of two other fraction.

During Jainism in the following field, Mathematics was developed

i) Theory of Numbers
ii) Arithmetical operations
iii) Geometry
iv) Operations with fractions
v) Simple equations
vi) Cubic Equations
vii) Quadratic Equations
viii) Formula for π
ix) Operations with logarithms
x) Sequences and progressions
xi) Permutations and combinations

The Bakhshali Manuscript (200 BC-200AD)

The Bakhshali manuscript was originally discovered in 1881, by a farmer working in the field of a village named Bakhshali (which is in Pakistan now). The Bakhshali manuscript was written on leaves of Birch in Sarada charecters and Gatha dialect which is a combination of Sanskrit and Prakrit.

Regarding the time period of original work of Bakhshali Manuscript there are different opinions, which are given in detail by Ian G Pearce (2002).

During this period in the following field, Mathematics was developed in the Bakhshali Manuscript:

i) Arithmetic and Algebra mainly
ii) Examples of the rule of three (profit, loss and interest)
iii) Solution of linear equations with as many as five unknowns
iv) The solution of the quadratic equations
v) Arithmetic and Geometric progressions
vi) Compound series
vii) Quadratic indeterminate equations
viii) Simultaneous equations
ix) Fractions and other advances in notations including use of zero and negative sign
x) Improved method for calculating square root and hence approximations for irrational numbers.

(200-400AD)

During the period from 2nd to 4^{th} centuries Ad, due to huge political revolutionary change, the subject couldnot be flourished much, though the practice of writing Siddhantas (astronomical works) which had started around 500 BC continued.

Of the Siddhantas, the Pitamaha Siddhanta is the oldest (500BC) and the Surya Siddhanta (400 AD) author unknown is the best known, which influenced Aryabhatta-I.

The major contribution of these work in Siddhantas was the invention of sine function.

CLASSICAL PHASE (400-1200 AD)

Following the establishment of "Gupta Dynasty" a galaxy of Mathematician - Astronomers led by Aryabhatta-I developed many areas of Mathematics and this period was known as Classical period.

The classical period of mathematics was known as the golden era in the history of Indian mathematics. Mathematcians like Aryabhatta I (476 AD- 550 AD), Varahamihira (505 AD -587 AD), Brahmagupta (598 AD-), Bhaskara I (c 600-680 AD), Mahavira(900 AD), Aryabhatta II (c 920-1000 AD) Bhaskara II or Bhaskaracharya, (1114 AD-), Madhava of Sangramagrama had immense contributions.

Their works are described below according to the chronological order.

Aryabhatta I (476AD-550AD)

Aryabhatta's contribution in the history of Indian mathematics is incomparable. The great Indian mathematician did more than just to give the world 'zero'. One of his major contributions is his book "Aryabhatiya", which was a compilation of mathematics and astronomy. It is a brief descriptive work having 123 metrical stanzas,consisting of supplementary rules of calculations in astronomy and mensuration. It contains several branches of mathematics such as algebra, arithmetic, plane and spherical geometry. It also includes the works on continued fractions, sum of power series, sine tables and quadratic equations.

a) The area of the triangle was given correctly as the product of half of the base and the altitude and the volume of the pyramid is also taken as the half of the product of the base and the altitude.
b) The area of the circle is found correctly as the product of the circumference and half of the diameter but the volume of the sphere is incorrectly as the product of area of a great circle and the square root of this area.
c) The area of a trapezium is stated as half of the sum of the parallel sides multiplied by the perpendicular between them.
d) One of the important statement written in the book Aryabhatiya is as follows:-

"Add 4 to 100,multiply by 8, and add 62,000.The result is approximately the circumference of a circle of which the diameter s 20,000."

Thus he demonstrated solutions to simultaneous quadratic equations and produced an approximate value of pi equivalent to 3.1416 correct to four decimal places. On the other side, Aryabhatta also used the value of 10 for pi, which was frequently used in India and is popularly known as Hindu value.

e) He used to estimate the circumference of the earth arriving at a figure of 24,135 miles only 70 miles of its true value.

f) In algebra, he provided the significant result for the summation of the series of square and cubes as follows:-

$1^3 + 2^3 + 3^3 + \dots\dots\dots + n^3 = n^2 (n+1)/4$
$1^2 + 2^2 + 3^2 + \dots\dots\dots + n^2 = n (n+1)(2n+1)/4$

g) Some trigonometric results were also there in Aryabhatiya. The sines of the angles upto 90 degree are given as 24 equal intervals of 3 3/4 degree. The radius was taken as 3,438 and the circumference as 360 X 60=21,600, in order to express the arc length and the sine length in the same unit.

For the sine table, the Hindus used a recursion formula that may be expressed as follows:-

If the n^{th} sum of the sequence from n=1 to n=24 is denoted by s_n and the sum of the first n

sines is S_n then $S_n + S_1 + S_n/S_1 = S_{n+1}$

So, we found the value of the modern trigonometric results as:-

$\sin 7\ 1/2° = 449$
$\sin 11\ 1/4° = 671$
$\sin 15° = 890$
$\sin 90° = 3{,}438$

The table also includes the values of the versed sine of the angle i.e 1- cosθ, from verse 3(3/4)° = 7, verse 90° = 3,438, if we divide the values of the tables by 3,438, then we get the similar to the modern trigonometric results.

Thus the introduction of an equivalent of the sine function, which replaces the Greek table of chords, is an important contribution.

Varahamihira (499AD-587AD)

Varahamihira was an Indian astronomer, mathematician and astrologer lived in Ujjain. His main work was the book, Pancasiddhantika, dated in 575AD. It was a treatise on mathematical astronomy and it summaries the five astronomical treatise, named, Surya Siddhanta, Romaka Siddhanta, Paulisa Siddhanta, Vasisha Siddhanta and Paitimaha Siddhanta. The Surya Siddhanta or the system of the Sun was written about 400AD. It is considered that the Surya siddhanta consists of certain astronomical rules n Sanskrit verses, with little explanation and without proof. The Paulisa Siddhanta was written about 380AD. The Romaka siddhanta and the Paulisa Siddhanta were works of western origin which influenced Varahamihira's thought and he summarized it. Varahamihira was involved more in astronomical works than in mathematics, which is evident from his works and treatises.

Some mathematical contributions of Varahamihira are:

♦ In trigonometry, Varahamihira improved the accuracy of sine tables of Aryabhatta. Some important results given by Varahamihira are:-

$\sin(x) = \cos((\Pi/2)-x))$

$\sin^2 x + \cos^2 x = 1$

$(1 - \cos 2x)/2 = \sin^2 x$

♦ He was the first mathematician to discovered a version of Pascal's triangle ,which he used to calculate the binomial co-efficients.

Brahmagupta(598AD-668AD)

Brahmagupta is a well known mathematician, flourished during 628 AD , more than a century after Aryabhatta. His major works includes the book "Brahmasphutasddhanta" which contains many useful results frequently used today. These are:-

It stated:-

' A positive number multiplied by a positive number is positive.'
' A positive number multiplied by a negative number is negative.'
A negative number multiplied by a negative number is positive.'

It contains many geometric results like the pythagoras theorem for right-angled triangle.

His mathematical inventions are as follows:-

♦ He mentions two value of pi-one is the practical value 3, and the other is the neat value 10 , but not more accurate than that of Aryabhatta.

♦ He found the ' groos' area of an isosceles traiangle by multiplying half of the base by one of the equal sides; for the scalene triangle with base 14 and sides 13 and 15, he found the gross area by multiplying half of the base by the arithmetic mean of the other sides.

♦ To find the exact area he used the Archimedean-Heronian formula. He gave the equivalent of the correct trigonometric results $2R = a/\sin A = b/\sin B = c/\sin C$ for the radius of the circle circumscribed about a triangle. But this is just a form of the wellknown results of Plotemy in the language of chords.

♦ The most appreciative work of Brahmagupta is his generalization of Heron's formula which is stated as

$K = \sqrt{(s - a)(s - b)(s - c)(s - d)}$ where a,b,c,d are the sides and s is the semiperimeter. But the formula was applicable only for the cyclic quadrilaterals. The correct for an arbitrary quadrilateral is given by $K = \sqrt{(s - a)(s - b)(s - c)(s - d) - abcd \cos^2\alpha}$.

He gave the pre-Hellanic formula as a rule for the gross area of a quadrilateral, which is stated as the product of the arithmetic mean of the opposite sides.

Brahmagupta's contribution to algebra is of much higher order than that of his rules for mensuration. The systematized arithmetic of negative numbers and zero was found in his work.

One of his important works is the use of the zero in the mathematical calculations. He established the basic mathematical rules for dealing with zero 1+0=1; 1-0=1; 1*0=0 and the negative numbers.

♦ He pointed out that quadratic equations could have two solutions, onw of which may be negative.

♦ It was mentioned that the Hindus regarded the irrational numbers as numbers, which was of great help in algebra and Indian mathematicians are praised for this. One of his noteworthy contribution includes that he found a rule for the formation of the Pythagoran triads expressed in the form $m, 1/2((m^2/n)-n)), 1/2((m^2/n)+n))$, which is a form of the old Babylion rule.

Brahmagupta proposed a theorem known as:

"If a cyclic quadrilateral is orthodiagonal (i.e has perpendicular diagonals), then the perpendicular to a side from the point of intersection of the diagonal always bisects the opposite side".

He was first to give a general solution of the linear Diophantine equation $ax + by = c$, where a,b,c are integers .For this equation to have solution, the gcd of a and b must divide c. Brahmagupta knew that f a and b are relatively prime, then the solution of this equation is given by $x = p + mb$, $y=q-ma$.

Bhaskara I(600AD-680AD)

Bhaskara was a great mathematician of 7^{th} century, who was popularly known for hid sine approximation formula. He was a great follower of Aryabhatta and one of the renowned scholars of Aryabhatta's astronomical school. He was the first write numbers in the Hindu decimal system with a circle for the zero. He wrote a commentary on Aryabhatta's work, named Aryabhatiyabhasya, in 629AD, in which he gave a remarkable approximation for the sine function. This commentary is one of the oldest work in Sanskrit on mathematics and astronomy.

The representation of numbers in a positional system is one of his most praiseworthy work. But the numbers were not written in figures, but in words or allegories, and were organized in verses. The numbers were represented as :- number 1 was given as moon, number 2 was given by wings, twins or eyes as they always occur in pair and number 5 was given by the five senses. These words were aligned in such a way that each number assigns the factor of the power of ten corresponding to its position in the reversed order: the highest powers were right from the lower ones. This system is well positioned as same words representing 4 can be used to represent 40 and 400. He was the first to use the Brahmi numerals in scientific way in Sanskrit. He explain a number given in a system , with the first nine Brahmi numerals, using a small circle for the zero.

His other works are: **Mahabhaskariya and Laghubhaskariya**.

Mahabhaskariya consists of 8 chapters dealing with astronomy. In chapter 7 of this book, Bhaskara suggested an approximate value of sine, which was unbelievably equal to accurate value of sine. The formula is given by:-

$\sin x = 16x(p-x)/5p^2 - 4x\,(p-x)$.

The approximate value of p was considered to be 10 for many centuries. But, according to Bhaskara, p had an irrational value, which was proved to be true later. Moreover, relations between sine and cosine, as well as between the sine of an angle >90 degree, >180 degree or >270 degree to the sine of an angle <90 degree are given are given.

Bhaskara had already stated the well- known Wilson's theorem, if p is a prime number, then 1+(p-1)! Is divisible by p. It was later proved by Al-Haitham and Fibonacci.

He states the theorems on the solutions of the Pell equation, $8x^2 + 1 = y^2$. He gave a simple solution to this, as, x=1, y=3, whose further solution can be constructed as (x ,y)=(6, 17).

ARYABHATTA II(920AD-1000AD)

Aryabhatta II was an Indian mathematician and astronomer. His most eminent works include, Maha-Siddhanta. It contains 18 chapters which are all written in verses in Sanskrit.

The first 12 chapters contains mathematical astronomy and covers the topics such as:-

- Longitudes of planets
- Lunar and solar eclipses
- The estimation of eclipses etc.

The next 6 chapters deals with:-

- Geometry
- Geography
- Algebra,which was used to calculate the longitudes of planets.

His other works include

- He elaborated the rules for solving intermediate equation given by:- by=ax+c

- He deduced a method to calculate the cube root of a number which was already given by Aryabhatta I.

- His one of the praise-worthy contribution includes his construction of a sine table which is correct upto 5 decimal places.

Bhaskara II (1114AD-1185AD)

Bhaskara II was one of the leading mathematician of the 12th century, whose work represents the culmination of earlier Hindu contributions. He filled the incomplete gaps of Brahmagupta's work. His main work is "Siddhanta Shiromani", which is divided into 4 parts, namely Lilavati, Bijaganita, Grahaganita, Goladhyaya.

The first section Lilavati, is named after his beloved daughter. It consists of 277 verses. It deals with the problems of calculations, progressions, measurement, permutations and many other topics. But Lilavati fails to distinguish between the exact and approximate statements for the area of a circle as, onequarter the circumference multiplied by the diameter and the volume of a sphere as one-sixth the product of surface area and diameter or ratio of circumference to diameter, he suggests either 3927 or1290 or the gross value 22/7. For the Pell's equation, which was proposed earlier by Brahmagupta. Bhaskara gave particular solutions for the five cases i.e p=8,11,32,61,67.

The second section Bijaganita has 213 verses, which discusses zero, infinity, negative and positive numbers and indeterminate equations including Pell's equation, solved using a Kuttaka method. He solved the equation, $61x^2 + 1 = y^2$ which was solved by Fermat and European contemporaries centuries later. Here we find the first statement stating that quotient is infinite. But it is not clearly understood as he stated a/0=0. Later he explained the previously misunderstood operation of division by zero. He found that dividing one into two pieces gives half i.e 1/2, so 1÷1/2 = 2 similarly, 1÷1/3 = 3. Thus dividing the smaller fractions yields larger numbers. Therefore, dividing one into pieces of zero size would yield infinitely many pieces, thus gives 1÷0 = inf inity.

The 3rd section, Grahaganita describes the instantaneous speed of the planets while treating their motion. Bhaskara also states that at its highest point a planet's instantaneous speed is zero. He arrived at a approximation:

$\sin y' - \sin y \approx (y' - y) \cos y$ for y' close to y in modern mathematics.

Madhava(1340AD-1425AD)

Madhava was a great mathematician and astronomer, lived in the town of Sangamagrama of Kerela. More popularly, he is known as the founder of the Kerela school of Mathematics and Astronomy, which of considered as the crown-jewel of Indian mathematics. He made great contribution in the field of trigonometry, geometry, algebra and calculus.

The Kerelease school has made astonishing achievements in series of expansions and geometries, arithmetic and trigonometric procedures, as well as astronomical observations. His most famous mathematical achievement are the Madhava-Leibnitz series for Π/4 and Madhava-Newton power series for the Sine and Cosine, but this work now survives only in a few verses recorded by later members of his school, as it is written in Mathematics in India by Kim Plofker.

Madhava dealt with the computation of pi, the ratio of the circumference of the circle and the diameter with the determination of the arbitrary arcs from their sines and cosines. The verse named Kriya-Kramakari on the Lilavati stated that, Madhava's value 2,827,433,388,233/900,000,000,000 is more accurate than the traditional value of 355/113. Madhava's value of pi is accurate to the eleventh decimal place as 3.14159265359.

Madhava then prescribes a geometric method for computing the value of a circumference of a circle by means of polygons. There is a set of four verses as cited in Kriya-Kramakari, which involves calculating the perimeters of successive regular circumscribed polygons, beginning with a square.

An alternative rule for the circumference is then described by Sankara, which corresponds to finding a sum of infinite number of terms in a series, and adding a final correction term, depending on the diameter D as follows:-

$C \approx (4D/1)-(4D/3)+(4D/5)-......(-1)^n\,(4D/2n-1) + (-1)^n\,(4D/2n^2-1)$

The final term $4Dn/(2n)^2 + 1$ in the series C compensates for the inaccuracy caused by the slow convergence of its terms.

Another mathematician Nilkantha (15th century, Tirur, Kerela) extended the results of Madhava elaborately, while Jyeshthadeva (16th century, Kerela) provided detailed proof of the theorems and derivations of the rules contained in the works of the Nilkantha and Madhava.

Another important discovery by Kerela Mathematican include the Newton-Gauss interpolation formula, the formula for the sum of an infinite series and a series of notation for pi.

Madhava's arc-difference rule is another procedure for finding an unknown arc from known sines and cosines, which is explained in the verses of Nilkantha's second chapter of Tantra-Sangraha itself which also contains a rule for computing the sine and cosine. The formula for the arc is stated as

"The divisor from the sum of the cosines is divided by the difference of the two given Sines. The radius multiplied by two is divided by that. That is the difference of the arcs." If the sine and cosine of the arc θ are known and the unknown θ + Δθ is known, then the difference Δθ is found from,

$$\Delta\theta \approx \frac{2R}{\dfrac{Cos\theta + Cos(\theta+\Delta\theta)}{Sin(\theta+\Delta\theta) - Sin\theta}}$$

Discussions

The remarkable developments in the field of Mathematics during the period of 17th century and 18th Century are the scope of the future works where the works of the great Mathematician Sri Srinivasa Ramanujan are most important.

The book "Studies in the History of Indian Mathematics" which is proceedings of a seminar on "the history of mathematics" held at the Chennai Mathematical Institute edited by C. S. Seshadri: Chennai Mathematical Institute, Tamil Nadu, India and a publication of Hindustan Book Agency, will serve the dual purpose of bringing to the international community a better perspective of the mathematical heritage of India and conveying the message that much work remains to be done, namely the study of many unexplored manuscripts still available in libraries in India and abroad.

The topics covered include:

(1) geometry in the Sulvasåtras;
(2) the origins of zero (which can be traced to ideas of lopa in Pàõini's grammar);
(3) combinatorial methods in Indian music (which were developed in the context of prosody and subsequently applied to the study of tonal and rhythmic patterns in music);
(4) a cross-cultural view of the development of negative numbers (from Brahmagupta (c.628 CE) to John Wallis (1685 CE);
(5) Kuññaka, Bhàvanà and Cakravàla (the techniques developed by Indian mathematicians for the solution of indeterminate equations);
(6) the development of calculus in India (covering the millennium-long history of discoveries culminating in the work of the Kerala school giving a complete analysis of the basic calculus of polynomial and trigonometrical functions);
(7) recursive methods in Indian mathematics (going back to Pàõini's grammar and culminating in the recursive proofs found in the Malayalam text Yuktibhàùà (1530 CE)); and
(8) planetary and lunar models developed by the Kerala School of Astronomy.

The articles in this volume cover a substantial portion of the history of Indian mathematics and astronomy.

Conclusion

In the present article, the main objective is to provide a chronological flowchart for the developments of the Indian Mathematics during ancient and medieval periods.

Mathematics on the Indian subcontinent has a rich history over 5000 years. The significant contribution of the invasion of zero to the developments made by Ramanujan is praise worthy. Several praise worthy results and rules which are widely used in the practical application of the mathematical works today, were discovered by the ancient great mathematicians of India long before their corresponding discovery in the middle east and western countries. From the paper of Mishra, it is found that until the last of 18th century, Europeans were unaware of the Indian culture and its ancient language, Sanskrit. By that time history was known to people only through the written documents provided by the Greek and the Latin authors. As a consequence the contributions of the Indian mathematicians remained uncovered. Another factor which resulted in this unawareness was the lack of proper written documents of the mathematical works discovered and also lack of proper education of the ancient people.

One of the administrator from the East India Company named Charles Wilkins (1749-1836) learned Sanskrit somehow and along with the Bengali pundits helped Sir William Jones to learn the language. After that, Jone's took initiative and the Asiatic Society of Bengal was found on the first day of 1784 with Jones as the president. The journal of the society took measures to reveal India's past and in November 1784, the first direct translation from Sanskrit to English was made. In 1786, a Persian scholar published the translation of the four Upanishads from a 17th century Persian version containing 50 Upanishads. As a result of this translation, Europeans got interested in Sanskrit and thus the hidden treasures of the India's past was uncovered.

Thus it is possible only in attempting a new synthesis of the ancient civilization with the achievement of modern cultures.

References

[1]. Datta B. Ancient Hindu geometry: the science of the Sulba. Cosmo; 1993.
[2]. Needham J. Science and civilisation in China. Cambridge University Press; 1976.
[3]. Dharampal. Indian Science and Technology in the Eighteenth Century: Some Contemporary European Accounts. Impex India; 1971.
[4]. Gupta RC. The madhava-gregory series. Math. Ed. 1973;7:B67-70.
[5]. Gupta RC. Symposium on History and Philosophy of Mathematical Sciences. Historia Mathematica. 1979 Aug 1;6(3):320-2.
[6]. Van Sertima I. Blacks in science: Ancient and modern. Transaction Publishers; 1983.
[7]. Boyer CB. A History of Mathematics; With a Foreword by Isaac Asimov. revised and with a preface by Uta C. Merzbach, New York. 1989.
[8]. GUPTA RC. The Chronic Problem of Ancient Indian Chronology. Ganita Bharvati. 1990;12(1-2):17-26.
[9]. Shastri VV, Hankey A, Sharma B, Patra S. Impact of pranayama and vedic mathematics on math anxiety and cognitive skills. Yoga Mimamsa. 2017 Jul 1;49(2):53.
[10]. Joseph GG. The Crest of the Peacock, Non-European Roots of Mathematics. 1991.

[11]. Pearce IG. Early Indian culture-Indus civilisation. Indian Mathematics: Redressing the balance. 2002.
[12]. Teresi D. Lost discoveries: the ancient roots of modern science--from the Babylonians to the Maya. Simon and Schuster; 2002.
[13]. Tewari R, Kaisarbagh L. The origins of Iron-working in India.
[14]. Mishra V. Mathematical Heritage of India: Some Remedial Issues: Mathematics History in Teaching.
[15]. Plofker K. Mathematics and its worldwide history.